RESEARCH STATEMENT

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Over the last ten years, I have pursued a broad range of research topics. My main research interests are: discrete harmonic analysis, analytic number theory, and moments of arithmetic functions. I have completed research projects in ergodic theorem, primes distribution, partitions theory, and additive number theory. My notable results during my Ph.D. are ℓ^p-improving inequalities of several averages over prime numbers, reporting a cancellation phenomenon as well as its application in partition theory and distribution of prime numbers, and solving a statistical version of the 105 problem raised by Erdős and Graham. I am continuing to work on discrete harmonic analysis and additive number theory.

Contents

1. Main results in analytic and additive number theory

This section is devoted to exploring the results I proved along with my advisor, Ernie Croot. The cancellation result is in review (see the arxiv version in [6]). The Erdős-Graham open problem is in preparation and will be submitted within the next two months.

1.1. The Erdős-Graham open problem.

This project is joint work with my advisor, Ernie Croot, and my colleague Maxie Schmidt. We are preparing the paper to submit to a journal. Erdős, Graham Ruzsa, and Straus in [1] proved the following theorem:

Theorem 1.1. *For primes p, q, there are infinitely many integers n such that* $\gcd\left(\binom{2n}{n}, pq\right) = 1$.

The study of divisibility of central binomial coefficients is useful to explore the distribution of prime numbers. A few studies about these coefficients can be found in [2–5]. The next natural step is an open problem by Erdős-Graham-Pomerance.

Question 1.2. [2, The 105 problem] *Are there infinitely many integers n such that* $\gcd\left(\binom{2n}{n}, 105\right) = 1$*?*

A sufficient condition for $\gcd\left(p, \binom{2n}{n}\right) = 1$ is that when computing $n+n$ base p, there are no carries. That is writing the digits of n base p as $n_{p,1}, n_{p,2}, \cdots$, one has $n_{p,j} < \lfloor p/2 \rfloor$ for all j. In this light, the Erdos-Graham conjecture would be true if there are infinitely many integers with $n_{3,j} < 2$, $n_{5,j} < 3$, and $n_{7,j} < 4$ for all j. Let the valuation function $v_p(n)$ be the largest exponent of p that divides n, for example $v_5(75) = 2$. What we proved was a statistical case of a generalization of the 105 problem:

Theorem 1.3. *Let $p_1 < p_2 \cdots < p_r$ are primes larger than a univeral constant. There are infinitely many integers n such that*

$$v_{p_i}\binom{2n}{n} = o_{p_i,r}\left(\log_{p_i}(n)\right) \quad \textit{for } 1 \leqslant i \leqslant r.$$

Note that the little oh is just with respect to p_i, r and not n.

Date: October 15, 2021.

To save space, I will give a big picture of the solution without mentioning the detail. First, we changed the infinite essence of the problem by truncating the digits and using an iterative algorithm. The idea is to pick a block of digits in a base prime p and fix the digits of blocks in other bases p_i. For example, assume that we consider each digit as a block and we start from 1000000000000_3. This number is 114001231_5, which has $3, 4$ in its digits base 5. Then we do the following changes to get the desired integer both in bases $3, 5$.

$$\begin{aligned}
1000000000000_3 &= 114001231_5\\
1001000000000_3 &= 120113444_5\\
1001000100000_3 &= 120120432_5\\
1001000101000_3 &= 120121034_5\\
1001000101100_3 &= 120121103_5\\
1001000101110_3 &= 120121111_5
\end{aligned}$$

It turns out that we can perform this algorithm, if in each iteration with length of the block $k = o(\log\log\log\log(n))$, for almost all $\frac{1}{p_i} < \alpha_i < 1$ there exists integer m such that $\alpha_i m + \beta_i$ have small digits in base p_i. So the conjecture is reduced to an additive number theory problem. The problem is very hard, and we had to ignore a little "base$-p_i$ pollution" and arrive at a statistical version of the result. Here is the way we formulated the problem:

Theorem 1.4. *Let N be a large integer and p be a fixed prime number. For $1 \leqslant i \leqslant r$, define $\alpha_{i,j} = p_i^{\{j\frac{\log(p)}{\log(p_i)}\}-1}$. For all but at most ϵN integers $\ell \leqslant N$ the following holds: for arbitrary $\beta_1, \beta_2, \cdots \in [0, 1)$, there exists $n \leqslant p^{\epsilon k}$, such that $n\alpha_{i,\ell} + \beta_i$ have small digits base p_i.*

We used discrete Fourier transformation of a counting function like Roth or Sarkozy theorem. Like most circle methods, we need to separate high-frequency coefficients, which can be controlled more easily. If we have Schanuel's conjecture, the terms $\{\frac{\log(p)}{\log(p_i)}\}$ become independent over $\mathbb{Q}$, which makes the theorem easy to solve. We had, thus, two options here: either try to prove the independence result (and the analogous one for four or more primes); or assume potential dependencies exist and somehow work with them. We chose the second option. The way to bypass Schanuel's conjecture was to parametrize all the curves from the dependency and use similar ideas like Gower's methods to find a large enough uniform distribution in our construction. We needed to show a certain matrix has a positive determinant.

1.1.1. *Future directions.* Generally, solving a non-statistical result is much harder because we need to prove that there is not even a single bad event, and this claim requires having perfect control over the oscillation of the frequencies. However, A direction that might solve this case is to argue that having a bad case for one block can be avoided by changing the previous block slightly. A natural question here is to see how close one can get to Pomerance's heuristic.

1.2. **The mysterious cancellation phenomena.** Our motivation was to prove the Pentagonal Number Theorem analytically. The first two dominant terms of the Ramanujan-Hardy-Rademacher formula for the number of partitions of n is

$$P_2(x) := \frac{\sqrt{12}e^{\frac{\pi}{6}\sqrt{24x-1}}}{24x-1}\left(1 - \frac{6}{\pi\sqrt{24x-1}}\right).$$

Pentagonal Number Theorem gives us the next immediate, but nontrivial, corollary.

$$\sum_{G_n \leqslant x} (-1)^n e\; p_2(x - G_n) \ll \sqrt{xp_2(x)}; \tag{1.5}$$

Note that by using a probabilistic argument, we can see that the bound in (1.5) is expected to be around $p_2(x)$.

1.2.1. *Our results.* What we have discovered is that 1.5 is just the tip of the iceberg, and that there is a very general class of sums like this that are small - much smaller than one would guess based on a probabilistic heuristic. Roughly, we proved that for certain real number $c > 0$ there exist $w_c < 1$ such that

$$\sum_{f(n) \leqslant x} (-1)^n e^{c\sqrt{x - f(n)}} = e^{cw_c\sqrt{x}},$$

where f is a quadratic polynomial with positive leading coefficient. More precisely, we proved two sets of cancellation inequalities depending on wether c is a real or complex number. For $c \in \mathbb{C}$, having an upper bound for the sum is harder, as we have both the fast growth of exponential functions and the extra oscillation coming from the imaginary exponent. We count three applications for the cancellation theorems.

First, we showed that in the "Weak pentagonal number theorem," we can replace the partition function $p(n)$ with Chebyshev Ψ function. A weak version of what we proved can be written as

$$\Psi(e^{\sqrt{x}}) = 2\sum_{0<\ell<\sqrt{x}T/2}\left(\Psi\left(e^{\sqrt{x - \frac{(2\ell-1)^2}{T}}}\right) - \Psi\left(e^{\sqrt{x - \frac{(2\ell)^2}{T}}}\right)\right) + O\left(e^{(\frac{5}{6}+\epsilon)\sqrt{x}}\right) \quad \text{where } T := e^{\frac{2\sqrt{x}}{3}}.$$

One can see that the measure of the set

$$I := \bigcup_{0<\ell<\sqrt{xT}/2} \left(e^{\sqrt{x-\frac{(2\ell)^2}{T}}}, e^{\sqrt{x-\frac{(2\ell-1)^2}{T}}} \right)$$

is almost half of the length of the interval $[0, e^{\sqrt{x}}]$. We proved that the number of primes in I, with weight $\log(p)$, is half the number of primes with the same weight. This prime counting gives a stronger result than one would get using a strong form of the Prime Number Theorem and also the Riemann Hypothesis(RH), where one naively estimates the Ψ function on each interval. We used our cancellation formula to control the low-height zeroes of the Riemann zeta function and the Van der Corput bound for exponential sums combined with the Montgomery Mean-value theorem to control the high-height zeroes.

Second we got a generalization of the Pentagonal Number Theorem for several forms of the famous partitions. For example for the usual number of partitions $p(n)$ we have

$$\sum_{\ell^2<x} (-1)^\ell p(x-\ell^2) \sim 2^{-3/4} x^{-1/4} \sqrt{p(x)}.$$

Before I mention the third application, I would like to mention the Vinagradov mean value Theorem. Assume that $J_{n,k}(N)$ is the number of solution to the set of equations

$$\sum_{i=1}^n a_i^r = \sum_{i=1}^n b_i^r \text{ for } 1 \leqslant r \leqslant k \text{ and } \sum_{i=1}^n a_i^{k+1} \neq \sum_{i=1}^n b_i^{k+1} \text{ and } a_i, b_j < N.$$

There was a conjecture which states that

$$J_{n,k}(N) \ll_\epsilon N^\epsilon \left(N^n + N^{2n-\frac{1}{2}k(k+1)} \right).$$

This conjecture has been proven for $k = 3$ at [9] and was proven for $k \geqslant 4$ at [10]. For more information in this direction, please check the survey paper [11]. There is a pigeon-hole argument that proves the existence of a solution for the case $k = O(\sqrt{n})$; but for larger k, the bound above becomes very close to the number of trivial solutions. To my knowledge, there is no proof to show even the existence of a nontrivial solution outside of this range. In another application of our cancellation theorem, we gave a constructive answer to the following problem.

Question 1.6. *Let $0 < c = c(N, n, k) < 1$ be the smallest constant such that there exists increasing sequences of positive integers $\{a_i\}$ and $\{b_i\}$ and $a_i, b_i < N$ for $1 \leqslant i \leqslant n$ that do not overlap, i.e. $a_i \neq b_j$, such that for all $1 \leqslant r \leqslant k$,*

$$\sum_{i=1}^n a_i^r - b_i^r \ll N^{cr} \tag{1.7}$$

How small can we take c to be for various ranges of k and n?

A weak version of our result was

Corollary 1.8. *Equation (1.7) has a constructive solution for $c = \frac{1}{2}$ and $k \sim \frac{n^{6/7}}{\log(n)}$ and $N = 2n^2$ as follows. For $1 \leqslant i \leqslant n$*

$$a_i = 2n^2 - (2i-2)^2 \in \mathbb{N} \quad \textit{and} \quad b_i = 2n^2 - (2i-1)^2 \in \mathbb{N}.$$

1.2.2. *Future directions.* Numerical results show that we can get a much better cancellation result. We believe that

$$\sum_{\ell^2<x} (-1)^\ell e^{\sqrt{x-\ell^2}} = e^{o(\sqrt{x})}.$$

Making c in corollary 1.8 very small is also an interesting problem. Of course, the form of a_i, b_i has to be changed, but even finding a constructive solution for $k \sim n^\epsilon$ and $c = 0$ is currently considered a breakthrough result. Unfortunately, we could not iterate the technique we used to prove corollary 1.8, but it seems that if one can repeat that the process for a suitable form of a_i, b_i, one may get a good chance of finding a constructive solution on a broader range.

Another direction we can aim for is to find a general approximated pentagonal number theorem for a broader range of partition functions, which is a viable project. As for the prime, I am curious about studying the combinatorial aspects of the "prime pentagonal number theorem" because we have a small error term for the number of primes in very short intervals. It would be an excellent project to try different Sieve methods or probabilistic combinatorics methods to examine our prime counting result.

2. Main results in discrete harmonic analysis

This project was started when I asked Michael Lacey to give me a problem to work on during a semester in which I took a course in harmonic analysis. Before explaining my projects in this area, let me briefly outline a few main results. We can start with Birkhoff Ergodic Theorem. A reference on this topic can be [13]. The discrete form of it would be that for a function $f \in L^1(\mu)$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(x-k) \text{ converges } \mu-\text{almost everywhere.}$$

After this result, Bourgain started the discrete generalized harmonic analysis field at 1980's by proving the ergodic theorem along the square integers. Today, it is a vibrant field with several recent important results.

He generalized the Birkhoff Ergodic Theorem for the square (and polynomial) jumps.

$$\frac{1}{\sqrt{n}}\sum_{k<\sqrt{n}} f(x-k^2) \text{ converges } \mu-\text{almost everywhere.}$$

One can also get quantitative versions of these results improving or maximal inequalities. Bourgain's argument for the polynomials relied heavily on the positivity of the function f. Ionescu and Wainger proved a more general case for all $f \in L^1$ by partitioning the low parts differently (see [12]). Later on, Terence Tao was able to give a sharper bound on this theorem using the superorthogonality and combinatorial properties of sunflower sets (see [14]).

2.1. Results in ergodic theorem and current projects.

Instead of a set of squares, one can choose the set of prime numbers.

$$A_N f(x) := \frac{1}{N}\sum_{n<N} f(x-n)\Lambda(n).$$

The set of primes is "full dimensional", so one can see that in an appropriate sense, an ℓ^1 function is improved to an ℓ^∞ function. The precise result is

$$\| A_N f \|_\infty \ll N^{-1} \| f \|_1 \log^t (\| f \|_1) .$$

where $f \in L^1(\mu)$ is a positive function, and $t = 1$ if we assume the GRH, otherwise $t = 2$. The point is that the bound is scale invariant.

The argument entails some subtle variants of the approach initiated by Bourgain. In particular, the notion of smooth numbers is essential due to their multiplicative properties, which helps us better control the Ramanujan sums in the low part. Moreover, in the absence of GRH, a systematic analysis of the exceptional Dirichlet's characters comes into the proof.

Another important point of choosing prime numbers can be seen in the proof of the Vinogradov Three prime theorem, which amazingly took us a step closer to the proof of the Goldbach conjecture.

Theorem 2.1. [18] *Every large enough odd integer can be written as a sum of 3 prime numbers.*

This theorem (also known as the ternary Goldbach theorem) was proved for every integer by Helfgott in [19] recently. Moreover, Vinogradov gave proof that with at most X^ϵ exceptions, every even integer less than X can be written as the sum of two prime numbers. However, it is important to mention that a complete proof of the Goldbach conjecture needs the lower bounds of the averages we consider while focusing on arriving at sharp upper bounds.

Currently we are studying another ℓ^p−improving inequality regarding primes in the progression. It is a natural next step after the case for the set of prime numbers is closed in [20–22]. Assume that

$$A_{N,y,b} := \frac{y}{N}\sum_{\substack{n<N \\ n\equiv b \pmod{y}}} \Lambda(n) f(x-n).$$

We wanted to find the upper bound C_y in

$$\| A_{N,y,b} \|_{p'} \ll C_y N^{\frac{1}{p'}-\frac{1}{p}} \| f \|_p \text{ where } f > 0 \text{ and } 1 < p < 2$$

We believe that an upper bound for C_y is $y^{\frac{1}{p}-\frac{1}{p'}}$, which seems natural and would be the scale-invariant bound. We needed to further divide the low parts depending on some parameter related to y. We also need to use Gauss sums instead of Ramanujan's sum to compute the low part. I think we can prove the maximal inequality as well as fixed scale bounds.

2.2. Further directions.

There are various directions at this point in this area. A project that we plan to work on it is related to improving inequality over the set of Gaussian primes. It is another natural question one may need to answer after dealing with primes. The analogous of the problem for a higher dimensional case for the square set has been answered by Wainger (known as spherical bounds). Krause, Mirek, and Tao introduced this problem in [15] as one of the possible open problems. We are interested in proving a quantitative inequality for

$$A_N f(x) = \frac{1}{N^2}\sum_{N(n_1),N(n_2)<N} \Lambda_{\mathbb{Z}[i]}(n_1)\Lambda_{\mathbb{Z}[i]}(n_2) f(x-n_1-n_2).$$

We have a good chance of proving at least an average version of it (with more logarithm weight functions) because it is known that Gaussian primes have uniform distribution both in magnitudes and in-phase (see [16]). It means that we can use Abel's summation formula twice and approximate A_N with a known arithmetic multiplier. The main difficulty arises because Ramanujan sums for higher-dimensional cases do not have the same cancellation properties as the one-dimensional case. That is the main restriction that makes us use more logarithm weight functions. I would like to study improving inequalities in more general quadratic number fields. The ultimate goal of this direction is

Question 2.2. *Let $\mathbb{F}$ be a quadratic number field and $\mathcal{O}_{\mathbb{F}}$ is its ring of integer. How many prime elements are needed to represent an element $x \in \mathcal{O}_{\mathbb{F}}$?*

Of course, this needs a lot of number theoretic and analysis supports, and some of the requirements may not be available at this time.

Checking the endpoint case for the primes in the progression might be another exciting project. I expect this problem to be extremely hard because we barely could control Ramanujan cancellations using smooth numbers. Nevertheless, we do not have tools like Cohen's identity for the Gauss sums, complicating the situation.

One other direction could be to study the bilinear case for the primes. We can consider Lacey's work at [23] as the first progress. Recently, Mirek, Krause, and Tao answered a conjecture in [15] about the bilinear ergodic theorem. A weakened version of the result states that for $\frac{1}{p} + \frac{1}{q} \leqslant 1$ and $f \in L^p$ and $g \in L^q$

$$A_N fg(x) := \frac{1}{N} \sum_{n<N} f(x-n)g(x-n^2) \text{ converges almost everywhere.}$$

An interesting problem could be one of the natural next steps after the new bilinear breakthrough, which could be considered state of the art in discrete harmonic analysis.

$$A_N fg(x) := \frac{1}{N} \sum_{n<N} \Lambda(n) f(x-n)g(x-n^2).$$

Although there are many more cases to explore, I will only mention one more possible direction related to our recent cancellation theorem. Inspired by the cancellation section, I am curious to study the sum

$$A_N f := \frac{1}{W(N)} \sum_{n<N} f(x-n)e^{\sqrt{x-n^2}} \text{ for a proper wieght } W(N).$$

We have Vander Corput theorem to study the minor arcs for the $\widehat{A_N}$ and our result to study the major arcs.

3. Brief Overview of Other Results and Projects

During my time at Georgia Tech, Sabanci University, and Tarbiat Modearres University, I had the privilege of collaborating with several people in analytic number theory, theory of partitions, coding theory, and cryptography and publishing more than ten research papers. Doing many different research projects gave me the confidence to study the related areas close to my primary interests. Another good point about all the projects and papers that I have worked on is that I now have many directions that are worth exploring. The variety of these open problems are suitable for an expert researcher to have breakthrough results, or even as an undergraduate research problem for my mentees. Most importantly, this subject has been my passion all these years, and I am looking forward to devoting my academic life to investigating new problems in number theory and its related areas. In this section, I explain my other results and a few interesting open problems very briefly.

3.1. Theory of partitions, $q-$series, and modular forms.

I explain two notable projects in this area. One has been published in the Ramanujan journal (see [24]), and the other one is almost finished and I will submit it in the near future.

3.1.1. *Number of partition with parts $pt + a$.* One of the many goals in this subject was finding the exact formula for partition functions with certain analytic properties like restricted partition, colored partitions, and partitions with parts in arithmetic sets. The classic technique to compute these functions is to use the modularity property of the generating function and couple it with the circle method. There are results about the exact formula of the partition functions with parts in arithmetic *symmetric* progression like in [25, 26] (i.e. if $x \in S$, then $M - x \in S$). In my Master's thesis at Sabanci University we aimed to prove the number of partitions with parts of the form $pt + a$ (p is a prime number) by controlling the growth of certain incomplete Kloosterman's sums. Completing the proof is a good project. Also, finding a modern proof like the one in [27] can be an impressive result. It is already known that the nonholomorphic modular forms have a term involving Kloosterman's sum in their $q-$series.

3.1.2. *Lambert series and Factorization Theorems.* This is joint work with my colleague, Maxie Schmidt. We built on and generalized recent work on so-termed factorization theorems for Lambert series generating functions. These factorization theorems allowed us to express formal generating functions for special sums as invertible matrix transformations involving partition functions. We could prove

$$\sum_{n\leqslant x} f(d) = \sum_{d|x+1} s_{x,d} \left[\sum_{j=0}^{d} \sum_{k=1}^{j-1} \sum_{\substack{0\leqslant i\leqslant j \\ j-k-G_i\geqslant 1}} (-1)^{\lceil \frac{i}{2} \rceil} p(d-j)\chi_k(j-G_i) \cdot f\left(\frac{(x+1)k}{d}\right)\right],$$

where χ_k is the principal Dirichlet character modulo k and $G_j := \frac{1}{2}\lceil j/2 \rceil \lceil (3j+1)/2 \rceil$ denotes the generalized pentagonal numbers. The contribution of this method is that it reduces the Fourier computation into a linear algebra problem, and one can work with a large but explicit matrix to carry out complicated computations regarding the Fourier transform of the average sum. As is expected, we took the additive average of a function with multiplicative considerations and used additive tools like the Lambert series to end up with a cocktail of additive and multiplicative functions tied up together. It provided us interesting new identities of multiplicative functions (like Euler's totient function or divisor function) with respect to the additive functions (like partition function). I am interested in simplifying the above identity and possibly using it to find moments of the multiplicative functions.

3.2. **Wireless sensor networks, cryptography, and machine learning.** In this section, I explain all of my engineering-related research experience very briefly.

3.2.1. *Wireless sensor networks.* My master's thesis at the Shiraz University of technology was about WSNs, and we published two papers in this area (see [28, 29]). It is desired to reduce the power consumption of sensors while keeping the distortion between the source and its estimate at the fusion center (FC) below a specific threshold. Our model had several relay-FC connections, and we assumed that Sensors could estimate a time-varying Gilbert-Elliott channel and send their data to FC. We needed to mathematically find the probability density function of the least number of "necessary" sensors so that FC can decide how many sensors should be off based on this information. This work may be interpreted as a first-hit Markov chain problem. We could have an acceptable bit error rate while reducing the sensor's power more than the existing models at the time. One of the directions for the future is to try and use a deep learning algorithm to teach FC how to choose the sensors and reduce the computational complexity. Another possible idea to advance can be to work on the design we need for the structure of the nodes. It can be a good step towards a practical design for the mentioned system model, which frequently happens in different applications.

3.2.2. *Homomorphic cryptosystems in E-voting protocols.* The main idea of the ElGamal homomorphic cryptosystems is that the encryption of the tally of the addition of all the votes is equal to the tally of the encryption of the vote. So we can encrypt all the votes to blind them, add them up, and decrypt the sum to get the result. However, one of their main challenges is the limitation on the number of voters. Our approach was inspired by the fact that preparing a smaller space of possible voting results demands a lighter load of computations. We used parallel subprotocol with multidimensional ballots to implement the information of plaintext, which results in a smaller size of ciphertext (see our paper [30]). One direction can be applying the OFDM channels and MIMO settings to provide a more efficient and more secure connection. Another critical issue is making the protocol more practical even in areas not covered with a proper network connection. Also, from a mathematical point of view, the following question is crucial and can undermine the whole philosophy of the E-voting homomorphic protocols.

Question 3.1. *Let $(G, \cdot)$ be a group with primitive point P and order N. Assume that $n \cdot P$ is given where $0 \leqslant n < N$. Is it an NP-complete problem to find out whether $n < N/2$?*

Answering this question would be considered a breakthrough in the subject, although it would be very challenging, and the ingredients of the proof may not even be available currently.

3.2.3. *Machine Learning projects.* I studied machine learning recently, and I have worked on a couple of projects. One of the projects is about Speech emotion recognition (SER). In this work, we extracted both local and global time-frequency features and Mel spectrogram images, listed as the essential features containing emotional information. We explored several machine learning algorithms from traditional approaches such as Naïve Bayes and SVM to modern convolutional and artificial neural networks. The emotions like anger and calm were detected with high accuracy by all the classifiers, while SVM and 1D-ANN showed better performance in detecting sadness and happiness. Moreover, the ensemble classifier helped to reduce the overfitting and improve the test accuracy to 76%.

The second project was to evaluate retinopathy grade regarding eye fundus color numerical images using a biomarker. First, we preprocessed the images using resizing, uniform cropping, normalization, and improving contrast. Second, we applied traditional methods to extract the optical disks, exudates, blood vessels, and global features. Next, we fed the biomarkers to SVM and used a pretrained CNN to train and test the entire processed images. Our best ensemble performance was at 81%, which is a significant rate.

3.3. **Non-Commutative Algebra and Coding Theory.** I have a few published or unfinished works in non-commutative algebra (monoid rings, and skew power series) from the time I was at Tarbiat Modarress. As the skew polynomials play a critical role in coding theory, I also published a few papers in coding theory (see [32–34]). I will explain a few important items here. One of the projects I have completed focussed on the chain conditions on the principal ideas in a monoid ring. Mimicking the Hilbert theorem, there are several lifting theorems, which state if condition A is transferred from R to $R[x]$ or $R[[x]]$.

Theorem 3.2. *Let R be a $\alpha-$rigid (not necessarily commutative) ring satisfying the ACC on annihilators, and α be a surjective endomorphism. Then R is right Archimedean, and α preserves nonunits of R if and only if the reduced ring $R[[x;\alpha]]$ is right Archimedean.*

This result was published in the Journal of Algebra and Its Applications (see [31]). There is an open problem, which seems like a natural next step. To my knowledge, there is no proof for the question so far:

Question 3.3. *Let R be an $\alpha-$compatible ring satisfying the ACC on annihilators. Furthermore, let α be a surjective endomorphism in R. Then R is right (resp. left) Archimedean if and only if the reduced ring $R[[x;\alpha]]$ is right (resp. left) Archimedean.*

In another related project, we explored the module structure of a few group rings that are important in skew cyclic codes ($R := \mathbb{F}_p[x;\alpha]/\langle x^n - 1\rangle$ in particular). We found all generator polynomials for these codes and described their minimal spanning sets. Moreover, we presented an encoding and decoding algorithm for skew cyclic codes over the ring R. Finally, we constructed examples of the best codes or the optimal codes in the paper (see [32]). I plan to complete a project investigating the group ring of these codes at [35].

3.4. **Summary.** In summary, I have successfully completed numerous projects and published in a range of mathematical areas. My experience includes successful research in analytic and additive number theory, discrete harmonic analysis, and noncommutative algebra and coding theory. I look forward to an exciting research career in which I can collaborate and publish. I am looking forward to contributing to a vibrant math community in your department and in the broader fields of mathematics.

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